A WEIGHTED L_2 BASED METHOD FOR THE DESIGN OF ARBITRARY ONE DIMENSIONAL FIR DIGITAL FILTERS

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ABSTRACT

FIR filters obtained with classical L_2 methods have performance that is very sensitive to the form of the ideal response selected for the transition region. This is because, usually, filter specifications do not constrain in any way the ideal frequency response inside this region. In this paper we propose a new general method for the weighted L_2 based design of arbitrary FIR filters. In particular we propose a well defined optimization criterion that depends on the selection of the desired response inside the transition regions. By optimizing our criterion we obtain desired responses that produce weighted mean square error optimum filters with extremely good characteristics. The proposed method is computationally simple since it requires the solution of a linear system of equations.

1. INTRODUCTION

The design of one-dimensional (1-D) digital filters, although is an old problem with significant existing literature, it has been of growing interest over the last decade mainly because digital filters are widely used in a variety of signal processing applications.

The most well known class of 1-D FIR filters is the class of linear phase filters. Their popularity stems mainly from the fact that corresponding design methods involve only real functions which also allows for the successful employment of the L_∞ criterion, the most suitable one, for the filter design problem. Linear phase filters are known however to introduce significant delays when their lengths are large. In applications where long delays are unacceptable, it is clear that there is a need of alternative filters. Furthermore there are problems which are by nature nonlinear-phase such as, constant group delay FIR filters, FIR equalizers, beamformers, etc. These problems require a filter design methodology that is significantly different from the conventional used in linear phase. In particular one can no longer be limited to real functions and needs to take into account general complex filters. What constitutes the complex function design problem challenging, from a methodology point of view, is the lack of efficient L_∞ techniques as compared to linear phase where the Remez Exchange Algorithm is dominant. This is largely due to the non-existence of a suitable counterpart, in the complex case, to the Alternation Theorem [3] that can serve as a base for developing computationally efficient algorithms. Consequently L_∞ techniques rely on sophisticated and computationally intense optimization machinery [1], [2].

The L_2 criterion, in the case of constant weight, results in the well known Fourier series coefficients which, in most cases, can be easily obtained analytically. The poor performance of this classical method (due to the Gibb's phenomenon) can be improved by introducing transitions regions between the passbands and stopbands of the filter, an idea also used in the linear phase case. Therefore existing complex filter design techniques mainly become generalizations to their real linear phase counterparts In particular, two of the above methods, the "don't care", [4] and the eigenfilter [5] seem to have comparable performance outperforming the other techniques. Both methods succeed in reducing the Gibb's phenomenon, their performance on the other hand can be seen to be significantly inferior to the L_∞ optimum solution, whenever such filter is available. In [6] an L_2 based method suitable for the unweighted design of the zero phase FIR digital filters was introduced. The basic characteristic of this method is that it is capable of optimally defining the unknown part of the ideal frequency response inside the transition regions and drastically reduce the Gibb's phenomenon. The method outperforms the most popular non L_∞ design techniques while it compares very favorably with the actual L_∞ optimum solution. In this work we intend to extend this idea to the complex filter design case and also include variable weighting function. In the next section we are going to present the proposed filter design method.

2. OPTIMIZATION CRITERION AND OPTIMUM APPROXIMATIONS

It is well known that the filter design problem, when considered in the frequency domain, is equivalent to a function approximation problem. Since frequency responses are periodic function with period 2π we can limit ourselves to the frequency interval $[-\pi, \pi]$. Suppose that the complex function $d(\omega)$, defined on the interval $[-\pi, \pi]$, denote the desired frequency response. We like to approximate $d(\omega)$ using linear combinations of the complex exponentials $e^{jn\omega},\ n = N_1, \ldots, N_2$; with the coefficients of the combination constituting the filter coefficients.

In this work we consider only the case $N_2 - N_1$ being an even integer since the odd case can be treated similarly. Without loss of generality we can assume that $-N_1 = N_2 = N$. This is so because it can be proved [3] that, approximating $d(\omega)$ with linear combinations of $e^{jn\omega},\ i = N_1, \ldots, N_2$, is equivalent to approximating the complex function $d(\omega)e^{j0.5(N_1+N_2)\omega}$ with linear combinations of $e^{jn\omega},\ n = -(N_2 - N_1)/2, \ldots, (N_2 - N_1)/2$.

Let now $f(\omega)$, $g(\omega)$ be two functions defined on $[-\pi, \pi]$, we can then define their usual inner product as

$$< f, g >= \int_{-\pi}^{\pi} f(\omega) g^*(\omega) d\omega, \tag{1}$$

where superscript "*" denotes complex conjugate. Similarly if $\mathbf{f}(\omega) = [f_1(\omega) \; \cdots \; f_k(\omega)]^t$ and $\mathbf{g}(\omega) = [g_1(\omega) \; \cdots \; g_m(\omega)]^t$ are two vector functions then $< \mathbf{f}, \mathbf{g} >$ denotes a matrix of dimensions $k \times m$ defined as

$$< \mathbf{f}, \mathbf{g} >= \int_{-\pi}^{\pi} \mathbf{f}(\omega)\mathbf{g}^H(\omega)d\omega, \tag{2}$$

where the superscript "^{H}" denotes conjugate transpose (hermitian). Finally with the help of the inner product we can define the norm of a scalar function $f(\omega)$ as $\|f\| = \sqrt{< f, f >}$.

Consider now the following vector function of ω

$$\mathbf{e}(\omega) = [e^{-jN\omega} \; e^{-j(N-1)\omega} \cdots e^{j(N-1)\omega} \; e^{jN\omega}]^t, \tag{3}$$

where superscript t denotes transpose, if $\mathbf{h} = [h_{-N} \; h_{-N+1} \; \cdots \; h_{N-1} \; h_N]^t$ is the vector of filter coefficients then the filter frequency response can be written as $h(\omega) = \mathbf{e}^H(\omega)\mathbf{h}$. Our goal is to select the coefficient vector $\mathbf{h}$ so as the corresponding function $h(\omega)$ approximates a desired response $d(\omega)$ optimally.

2.1. The Class of the Candidate Desired Responses

In order to formulate our L_2 approximation problem let us first define the class of frequency responses $d(\omega)$ and weighting functions $w(\omega)$ we are interested in. Let $-\pi = \omega_0 < \omega_1 < \ldots < \omega_{2M-1} = \pi$, be $2M$ distinct points in the interval $[-\pi, \pi]$. Suppose $d(\omega)$ is a complex function defined as

$$d(\omega) = \begin{cases} d_i(\omega) & \omega \in [\omega_{2i}, \omega_{2i+1}], \quad i = 0, \ldots, M-1, \\ g_i(\omega) & \omega \in (\omega_{2i-1}, \omega_{2i}), \quad i = 1, \ldots, M-1, \end{cases} \tag{4}$$

and $w(\omega)$ a real positive function defined as

$$w(\omega) = \begin{cases} w_i(\omega) & \omega \in [\omega_{2i}, \omega_{2i+1}], \quad i = 0, \ldots, M-1, \\ v_i(\omega) & \omega \in (\omega_{2i-1}, \omega_{2i}), \quad i = 1, \ldots, M-1, \end{cases} \tag{5}$$

where $d_i(\omega), \; w_i(\omega)$ denote the parts of the frequency response and weighting function that are known (corresponding either to passbands or stopbands) and $g_i(\omega), \; v_i(\omega)$ the parts corresponding to the transition regions that are unknown. If $\mathcal{U}_i = [\omega_{2i}, \omega_{2i+1}], \; i = 0, \ldots, M-1$; $\mathcal{T}_i = (\omega_{2i-1}, \omega_{2i}), \; i = 1, \ldots, M-1$, and $\mathcal{U} = \cup_{i=0}^{M-1}\mathcal{U}_i$ and $\mathcal{T} = \cup_{i=1}^{M-1}\mathcal{T}_i$ then $d(\omega)$ and $w(\omega)$ are known on $\mathcal{U}$ and unknown on $\mathcal{T}$.

Our goal now is to properly exploit the unknown part of the desired response in order to come up with an efficient filter design method. Since the weighting function is also unknown inside the transition regions in order to facilitate our design we propose to extend $w(\omega)$ on each transition interval $\mathcal{T}_i = (\omega_{2i-1}, \omega_{2i})$ by using an exponential interpolation scheme of the form $w(\omega) = v_i(\omega) = \alpha_i e^{\beta_i \omega}$. The parameters $\alpha_i, \; \beta_i$ can be uniquely specified by assuring that the resulting $w(\omega)$ is continuous.

For the known parts of the desired response and the weighting function, that is, functions $d_i(\omega), \; w_i(\omega)$ we make the following assumption:

$\mathcal{A}$**:** The parts $d_i(\omega), \; w_i(\omega)$ of the desired response and the weighting function defined on the intervals $\mathcal{U}_i = [\omega_{2i}, \omega_{2i+1}], \; i = 0, \ldots, M-1$, are functions with first derivative and well defined left and right second order derivatives.

2.2. Optimization Criterion

Since the desired response $d(\omega)$ is not defined in the transition regions, by selecting the functions $g_i(\omega)$, we end up with different possibilities for the desired response $d(\omega)$. For each such selection there corresponds an optimum filter that minimizes the weighted mean square error criterion $\|wd - wh\|^2$. It is also clear that the filter that minimizes the weighted mean square error will depend on the specific selection of $d(\omega)$, let us therefore denote it as $\mathbf{h}_d$; furthermore the corresponding minimum weighted mean square error will also be a function of $d(\omega)$, that is,

$$\mathcal{E}_0(d) = \|wd - wh_d\|^2 \tag{6}$$

where the optimum filter coefficients $\mathbf{h}_d$will be given by

$$\mathbf{h}_d =< w\mathbf{e}, w\mathbf{e} >^{-1} < w\mathbf{e}, wd^* > . \tag{7}$$

Using (6), we can now propose a means to optimally define the desired response $d(\omega)$ by further minimizing $\mathcal{E}_0(d)$ with respect to $d(\omega)$, that is,

$$d_o(\omega) = \arg\min_d \mathcal{E}_0(d) = \arg\min_d \|wd - wh_d\|^2. \tag{8}$$

The solution of the optimization problem defined by (8) can be easily proved that yields, inside the transitions regions, as optimum desired response $d_o(\omega) = \mathbf{e}^H(\omega)\mathbf{h}_{dc}$, and as optimum filter coefficients

$$\mathbf{h}_{d_o} = \mathbf{h}_{dc} \tag{9}$$

where $\mathbf{h}_{dc}$ is the optimum don't care filter. The main drawback of the optimum don't care filter is the fact that the resulting optimum desired response is not necessarily continuous. In order to come up with an optimality criterion that can design filters with improved characteristics we extend the idea presented in [6] and propose the following optimality criterion

$$\mathcal{E}_1(d) = \|(wd)' - (wh_d)'\|^2 \tag{10}$$

where $h_d(\omega) = \mathbf{e}^H(\omega)\mathbf{h}_d$ is the frequency response of the filter $\mathbf{h}_d$ defined in (7). As in the case of $\mathcal{E}_0(\cdot)$, we now propose the following optimization problem

$$d_o(\omega) = \arg\min_d \mathcal{E}_1(d) = \arg\min_d \|(wd)' - (wh_d)'\|^2. \tag{11}$$

The existence of the derivatives in the criterion (10) immediately excludes discontinuous desired responses as having infinite weighted mean squared error. Without loss of generality we can therefore impose the following constraint on $d(\omega)$.

$\mathcal{C}$**:** The desired response $d(\omega)$ is a continuous function which, at each frequency ω has left and right derivatives.

Of course in view of Assumption $\mathcal{A}$ we need to apply Constraint $\mathcal{C}$ only in the transition regions paying special attention to the end points of each such interval.

2.3. Optimum Filter and Optimum Desired Response

We will now present, without proof, a theorem that gives necessary and sufficient conditions for the optimality of the desired response $d_o(\omega)$ and its corresponding filter $\mathbf{h}_{d_o}$.

Theorem: In order for $d_o(\omega)$ and its corresponding filter $\mathbf{h}_{d_o}$ defined by (9) to solve the optimization problem defined by (11) and (7) it is necessary and sufficient that the following ordinary differential equation is satisfied for $\omega \in \mathcal{T}_i, \; i = 1, \ldots, M-1$, inside each transition region,

$$[w(\omega)(d_o(\omega) - h_{d_o}(\omega))]'' = -w(\omega)\mathbf{e}^H(\omega)\mathbf{p} \tag{12}$$

$$\mathbf{p} =< w\mathbf{e}, w\mathbf{e} >^{-1} < (w\mathbf{e})', [w(d_o^* - h_{d_o}^*)]' > . \tag{13}$$

With the help of Theorem we can find a system of linear equations that solves the optimization problem defined by (11). A point worth mentioning is the fact that when the weighting function $w(\omega)$ is constant then, from (13), the quantity $< (w\mathbf{e})', [w(d_o^* - h_{d_o}^*)]' >$, using integration by parts, is equal to $w < \mathbf{e}'', d_o^* - h_{d_o}^* >= 0$, with the last equality being the result of the orthogonality principle. Therefore for $w(\omega)$ constant (12) is reduced to the differential equation of Theorem 1 of [6].

Let us now present the unknown variables and the corresponding linear equations needed for the solution of the optimization problem. Notice that we already have introduced two parameter vectors that are inter-related, namely the optimum filter coefficients $\mathbf{h}_{d_o}$ ($2N+1$ unknowns) and the auxiliary vector $\mathbf{p}$ defined in (13) which also contains $(2N+1)$ unknowns. If we now integrate twice the differential equation in (12) inside each transition region $\mathcal{T}_i,\ i = 1, \ldots, M-1$ we obtain

$$w(\omega)d_o(\omega) = w(\omega)\mathbf{e}^H(\omega)\mathbf{h}_{d_o} - \mathbf{f}_i(\omega)^H\mathbf{p} + \mathbf{c}^H(\omega)\mathbf{q}_i \qquad (14)$$

where $\mathbf{c}(\omega) = [\omega\ 1]^t$; $\mathbf{q}_i$ is a vector containing the two unknown parameters of the solution of the differential equation (12), that is, $\mathbf{q}_i = [q_i^1\ q_i^2]^t$ and finally

$$\mathbf{f}_i(\omega) = \int_{\omega_{2i-1}}^{\omega}\int_{\omega_{2i-1}}^{\tau} w(s)\mathbf{e}(s)dsd\tau. \qquad (15)$$

is the double consecutive integration of the function $w(\omega)\mathbf{e}(\omega)$ with $\mathbf{e}(\omega)$ defined in (3). The vector function $\mathbf{f}_i(\omega)$ can be easily evaluated when we use the exponential extension of the weighting function. Notice that with (14) we have introduce M additional parameter vectors $\mathbf{q}_i,\ i = 1, \ldots, M-1$, which corresponds to $2M-2$ additional unknown variables thus raising the total number of unknowns to $4N + 2M$. It is clear that we need an equal number of equations in order to produce the solution to the optimization problem.

The necessary equations can be obtained from (9) and (13) using (14) and by imposing continuity at the two end-points of each transition interval $\mathcal{T}_i = (\omega_{2i-1}, \omega_{2i}),\ i = 1, \ldots, M-1$ on the solution $w(\omega)d_o(\omega)$ of (14). Specifically, from (9) and using (14) we obtain a first set of $2N+1$ equations as follows

$$\mathbf{A}\mathbf{h}_{d_o} + \mathbf{B}\mathbf{p} + \sum_{i=1}^{M-1}\mathbf{C}_i\mathbf{q}_i = \mathbf{h}_{\mathcal{U}} \qquad (16)$$

where

$$\begin{aligned}
\mathbf{h}_{\mathcal{U}} &= < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, wd^*\mathbb{1}_{\mathcal{U}} > && (17)\\
\mathbf{A} &= < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, w\mathbf{e}\mathbb{1}_{\mathcal{U}} > && (18)\\
\mathbf{B} &= \sum_{i=1}^{M-1} < w\mathbf{e}\mathbb{1}_{T_i}, \mathbf{f}_i\mathbb{1}_{T_i} > && (19)\\
\mathbf{C}_i &= - < w\mathbf{e}\mathbb{1}_{T_i}, \mathbf{c}\mathbb{1}_{T_i} >,\ i = 1, \ldots, M-1. && (20)
\end{aligned}$$

and $\mathbf{h}_{d_o}$ is the optimum don't care filter.

Similarly using the definition of $\mathbf{p}$ from (13) and using (14) we obtain $2N+1$ additional equations

$$\mathbf{D}\mathbf{h}_{d_o} + \mathbf{E}\mathbf{p} + \sum_{i=1}^{M-1}\mathbf{F}_i\mathbf{q}_i = \mathbf{p}_{\mathcal{U}} \qquad (21)$$

where

$$\begin{aligned}
\mathbf{p}_{\mathcal{U}} &= < (w\mathbf{e})'\mathbb{1}_{\mathcal{U}}, (wd^*)'\mathbb{1}_{\mathcal{U}} > && (22)\\
\mathbf{D} &= < (w\mathbf{e})'\mathbb{1}_{\mathcal{U}}, (w\mathbf{e})'\mathbb{1}_{\mathcal{U}} > && (23)\\
\mathbf{E} &= < w\mathbf{e}, w\mathbf{e} > + \sum_{i=1}^{M-1} < (w\mathbf{e})'\mathbb{1}_{T_i}, \mathbf{f}_i'\mathbb{1}_{T_i} > && (24)\\
\mathbf{F}_i &= - < (w\mathbf{e})'\mathbb{1}_{T_i}, (\mathbf{c})'\mathbb{1}_{T_i} >,\ i = 1, \ldots, M-1. && (25)
\end{aligned}$$

Finally by imposing continuity on the solution $w(\omega)d_o(\omega)$ of (14) at the two end points $\omega_{2i-1},\ \omega_{2i}$ of each transition region $\mathcal{T}_i$, in order to satisfy Constraint $\mathcal{C}$, we obtain the following $2M-2$ equations

$$\mathbf{G}_i\mathbf{h}_{d_o} + \mathbf{H}_i\mathbf{p} + \mathbf{J}_i\mathbf{q}_i = \mathbf{s}_i,\ i = 1, \ldots, M-1, \qquad (26)$$

where

$$\begin{aligned}
\mathbf{s}_i &= [w(\omega_{2i-1})d(\omega_{2i-1})\ \ w(\omega_{2i})d(\omega_{2i})]^t && (27)\\
\mathbf{G}_i &= [w(\omega_{2i-1})\mathbf{e}(\omega_{2i-1})\ \ w(\omega_{2i})\mathbf{e}(\omega_{2i})]^H && (28)\\
\mathbf{H}_i &= -[\mathbf{f}_i(\omega_{2i-1})\ \ \mathbf{f}_i(\omega_{2i})]^H && (29)\\
\mathbf{J}_i &= [\mathbf{c}(\omega_{2i-1})\ \ \mathbf{c}(\omega_{2i})]^H && (30)
\end{aligned}$$

which constitutes the last set of equation raising the total number to the desired $4N + 2M$.

Summarizing: In order to solve the minimization problem in (11) we solve the system of equations defined by (16), (21) and (26) which yields the optimum filter coefficients $\mathbf{h}_{d_o}$, plus certain auxiliary quantities $\mathbf{p},\ \mathbf{q}_i,\ i = 1, \ldots, M-1$, that can be used to obtain the optimum desired frequency response $d_o(\omega)$ through (14).

3. DESIGN EXAMPLES

In this section we are going to apply our method to a filter design example and compare it to other existing filter design techniques. In particular we will apply our method to the design of weighted nearly linear phase lowpass filters; and compare it against the don't care method of [4], and the Complex Remez algorithm of [3] which is included in Matlab as the function *cremez.m*. For our comparison we are going to focus on the maximum weighted magnitude error e_m inside the passbands and stopbands of the filter, as well as the maximum group delay error e_τ in the passbands.

Consider the following specifications

$$d(\omega) = \begin{cases} e^{-j\frac{4N}{5}\omega}, & \omega \in [-0.1,\ 0.3] \\ 0, & \omega \in [-1,\ -0.18] \cup [.38,\ 1] \end{cases}$$

with the weighting function equal to 1 and $\sqrt{2}$ in the passband and the stopbands respectively.

In Table I we present the maximal magnitude error e_m, and in Table II the corresponding group delay error e_τ for the three methods and for different filter lengths. The proposed method performs always better than the don't care method. What is however more interesting is the fact that for filter lengths greater than 101 it also outperfoms the Complex Remez algorithm. It is notable the fact that this performance is obtained with a very low computational cost while Complex Remez, as we said, is computationally demanding and practically useless for lengths exceeding 151.

We obtained similar results in all other design examples we considered with different values of the cutoff frequencies as well as with different values of the weighting function.

	Proposed	Don't Care	C. Remez
$2N+1$	e_m	e_m	e_m
51	1.77×10^{-2}	3.29×10^{-2}	1.24×10^{-2}
61	9.60×10^{-3}	1.83×10^{-2}	5.75×10^{-3}
71	4.87×10^{-3}	9.62×10^{-3}	3.73×10^{-3}
81	2.70×10^{-3}	5.75×10^{-3}	1.63×10^{-3}
91	1.26×10^{-3}	2.86×10^{-3}	1.98×10^{-3}
101	7.16×10^{-4}	1.76×10^{-3}	6.29×10^{-4}
111	3.35×10^{-4}	8.75×10^{-4}	4.43×10^{-4}
121	1.93×10^{-4}	5.13×10^{-4}	4.88×10^{-4}
131	9.75×10^{-5}	2.71×10^{-4}	4.03×10^{-4}
141	5.01×10^{-5}	1.43×10^{-4}	8.33×10^{-5}
151	2.77×10^{-5}	8.25×10^{-5}	7.96×10^{-5}

Table I. Maximum magnitude approximation errors resulted from the weighted design of the filter by different methods and for different filter lengths.

	Proposed	Don't Care	C. Remez
$2N+1$	e_τ	e_τ	e_τ
51	9.27×10^{-1}	1.03×10^{0}	7.10×10^{-1}
61	6.84×10^{-1}	8.54×10^{-1}	6.99×10^{-1}
71	5.42×10^{-1}	7.39×10^{-1}	5.13×10^{-1}
81	3.23×10^{-1}	4.87×10^{-1}	3.43×10^{-1}
91	2.31×10^{-1}	3.91×10^{-1}	2.66×10^{-1}
101	1.35×10^{-1}	2.48×10^{-1}	1.46×10^{-1}
111	8.13×10^{-2}	1.66×10^{-1}	8.06×10^{-2}
121	5.04×10^{-2}	1.12×10^{-1}	5.00×10^{-2}
131	2.59×10^{-2}	6.27×10^{-2}	2.93×10^{-2}
141	1.62×10^{-2}	4.28×10^{-2}	1.61×10^{-2}
151	8.00×10^{-3}	2.27×10^{-2}	1.04×10^{-2}

Table II. Maximum group delay approximation errors resulted from the design of the filter.

In Figures 1 and 2 we present the approximation errors in the magnitude and the group delay for the proposed (solid), the don't care (dashed), and the Complex Remez (half-tone), for a filter of length 21.

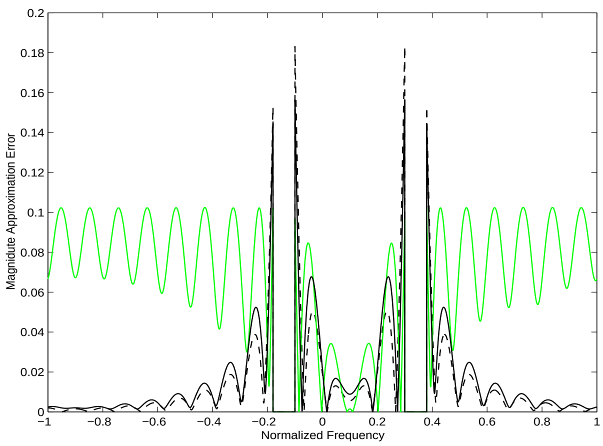

Figure 1. Magnitude approximation errors for the design of the nearly linear phase lowpass filter of length 21. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

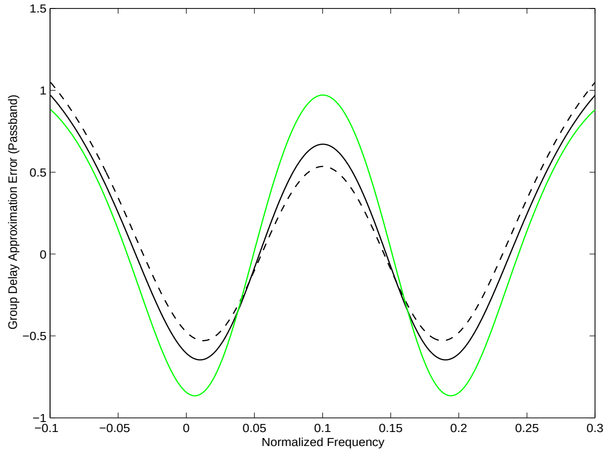

Figure 2. Group delay approximation errors inside the passband for the design of a nearly linear phase lowpass filter of length 21. Proposed method (solid), Complex Remez (halh-tone), don't care region (dashed).

4. CONCLUSION

We have presented a new L_2 based method for the design of arbitrary FIR digital filters. Minimizing a suitable L_2 measure results in an optimum extension of the ideal response inside the transition regions. The optimum filter is then obtained as the corresponding minimum weighted mean squared error filter. The complexity of our scheme is low since it requires the solution of a linear system of equations. In all design examples we carried out, our method always outperformed the don't care while, at the same time, it either compared favorably or even outperformed the computationally demanding Complex Remez algorithm.

5. REFERENCES

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